The Longest Run of Heads

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This paper is a review of older and recent results concerning the distribution of the longest head run in a coin tossing sequence, problem that arise in many applications and fields (such as quality control, realiability, etc.). First, we will describe exact and recurrent formulas for the distribution of the longest run, both in the case when the coin is fair ($p = q = \frac{1}{2}$) or biassed ($p \in (0, 1)$). We will observe that for long and very long sequences the exact formulas are not of much help and the need for bounds and asymptotic analysis is mandatory. The second section concerns with these problems. Finally we consider that is worth mentioning some extensions to other run related phenomena.

1. The exact and recurrent formulas for the distribution of the longest run of heads

We will describe in the following an interesting and simple classroom experiment, due to T. Varga as mentioned in Révész [12] or Schilling [13], from which the problem of finding the length of the longest pure heads in n Bernoulli independent trials arose naturally. The experiment begins by dividing a class into two groups. In the first group, students had to toss a coin 200 times (the coin is considered fair) and record the resulting sequence of heads and tails. In the second group, each student had to write down a *random simulation* of 200 coin tosses. After collecting the sheets of paper from the two groups, based on a simple argument, the students can be classified into their original groups with a surprising degree of accuracy. The argument is based on the following observation: in students *simulated* sequences, the longest run of the consecutive heads or tails is too short (maximum 5 units long) relative to that which arise from the natural coin tossing process (which is about 7 H or T long). The experiment described lead to a natural question: what is the expected value for the length of the longest head run in n tosses of a fair coin? We will try to answer this question and others in subsequent.

1.1. **The case of a fair coin.** In 1948, Bateman [1] gave an exact formula for the distribution of the longest run, in both fair coin or biassed coin cases. He started by considering a sequence of n elements (realizations of Bernoulli independent trials), $r_1 = k$ of which possessed a property E and $r_2 = n - k$ which don't posses that property (are $\bar{E}$), under the null hypothesis, H_0, that the elements of the sequence are in random order. His idea was to consider the partitions of r_1 and r_2 having s as the greatest part, where $s = 1, 2, \ldots, L$ and L is the given length for the longest run, and determine the different ways in which such partitions form $2t$ or $2t+1$ groups, where $t = 1, 2, \ldots, r_1 - L + 1$ for $r_1 \geq r_2$. He also exploited the fact that the number of partitions of $r_{i=1,2}$ into t parts of which the greatest part contains s elements can be expressed in terms of binomial coefficients, as this number represents the coefficient of x^{r_i} in the expansion $\left(x + x^2 + \cdots + x^s\right)^t = x^t \left(\frac{1-x^s}{1-x}\right)^t$. He obtained the following formula:

$$\mathbb{P}(L_n \geq m | k, n-k) = \frac{1}{C_n^k} \left[\sum_{j=1}^{\left[\frac{k}{m}\right]} (-1)^{j+1} C_{n-k+1}^{j} C_{n-jm}^{n-k} \right] \tag{1}$$

where L_n is the length of the longest run of heads in a sequence of n Bernoulli trials (coin tosses). He also obtained a formula for the distribution of L_n in the *biassed* case, i.e. the probability of head (or success) is $p \in (0,1)$, but this formula will be given in the next section.

On the same conditions, and using the observation that s failures divide the sequences of successes into $t = s+1$ runs and that the number of permutations with longest success run $< m$ is equal to the number of ways of selecting in order t integers whose sum is k, each between 0 and $m-1$, i.e. the coefficient of x^r in the expansion of $\left(1 + x + x^2 + \cdots + x^{m-1}\right)^t$, Burr and Cane [3] obtained the following formula for distribution of the length of the longest run:

$$\mathbb{P}(L_n < m | n, k) = \frac{1}{C_n^{n-k}} \left[C_n^{n-k} - C_{n-k+1}^{1} C_{n-m}^{n-k} + C_{n-k+1}^{2} C_{n-2m}^{n-k} - \ldots \right] \tag{2}$$

which is related to (1). They also gave a kind of recurrence relation useful in simulation (for moderate m):

(3)

$$\mathbb{P}(L_n < m | n, k) = 1 - \frac{t k^{(m)}}{n^{(m)}} \left\{ 1 - \frac{t-1}{2} \frac{(k-m)^{(m)}}{(n-m)^{(m)}} \left[1 - \frac{t-2}{3} \frac{(k-2m)^{(m)}}{(n-2m)^{(m)}} (1 - \ldots) \right] \right\}$$

where we used the notation $a^{(b)} = a(a-1)\ldots(a-b+1)$ if $a \geq b$ and $a^{(0)} = 0$ if $a < b$.

A recurrent formula for the distribution of the longest head run was obtained by Schilling [13] in both, fair or biassed coin, cases. Here we will describe the formula for the first case. If we denote the number of sequences of length n in which the longest run of heads does not exceed m, by $A_n(m)$, it is easy to see that the probability of length of the longest head run does not exceed m is equal with $\frac{A_n(m)}{2^n}$. The key to compute $A_n(m)$, as Schilling mentioned, is to partition the set of favorable sequences according to the number of heads that occur before the first tail (if any). Doing this we obtain that if $n \le m$ than $A_n(m) = 2^n$ and if $n > m$ each favorable sequence begins with one of the following subsequences: T, HT,..., $HH \dots HT$ (the last subsequence contain m heads) and is followed by a string having no more than m consecutive heads, which lead to:

$$A_n(m) = \begin{cases} \sum_{j=0}^{m} A_{n-j-1}(m), & \text{for } n > m \\ 2^n & \text{for } n \le m. \end{cases} \tag{4}$$

An interesting fact is that the number of sequences of length n that contain no consecutive heads, $A_n(1)$, is a Fibonacci number. Using this recursion formula we can easily compute the values for $A_n(4)$:

n	0	1	2	3	4	5	6	7	8	9	10	...
$A_n(4)$	1	2	4	8	16	31	61	120	236	464	912	...

so the probability that the length of the longest run is not greater than 4, in $n = 10$ coin tosses is $\frac{912}{2^{10}} = 0.890625$.

1.2. **The case of a biassed coin.** Now we will consider the case in which we have a sequence of n Bernoulli trials with the probability of success (or of head) to be $p = 1 - q \in (0, 1)$. As we mentioned in the previous case, Bateman [1] found a formula for the distribution of the length of the longest head run:

$$\mathbb{P}(L_n \ge m) = \sum_{j=1}^{\left[\frac{n}{m}\right]} (-1)^{j+1} \left(p + \frac{n - jm + 1}{j} q\right) C_{n-jm}^{j-1} p^{jm} q^{j-1} \tag{5}$$

The equation (5) was deduced from the equation (1) by:

$$\mathbb{P}(L_n \ge m) = \sum_{k=m}^{n} \mathbb{P}(L_n \ge m | k) \times \mathbb{P}(k) \tag{6}$$

and noticing that $\mathbb{P}(k) = C_n^k p^k q^{n-k}$ and $\sum_{k=0}^{n} C_n^k p^k q^{n-k} = 1$.
Using multinomial coefficients, Philippou and Makri [11] found exact formulas for the distribution of the number of head runs of length k, denoted by $N_n^{(k)}$, and for the length of the longest head run (L_n) in n independent coin tosses (Bernoulli trials). They observed, the same argument being found in Schilling [13], that a typical element ($N_n^{(k)} = x$) is an arrangement of the form:

$$a_1 a_2 \dots a_{x_1+x_2+\dots+x_k+x} \underbrace{ss\dots s}_{i}$$

for $0 \le i \le k-1$ and such that x_1 of the a's are T, x_2 of the a's are HT,...,x_k of the a's are $H\underbrace{T\dots H}_{k-1}$ and x of the a's are $\underbrace{H\dots H}_{k}$. Notice first that between x_1,...,x_k we have the relation $\sum_{j=1}^{k} jx_j + kx + i = n$ and that the number of such arrangements is given by the multinomial coefficient:

$$\binom{x_1 + \cdots + x_k + x}{x_1\ ,\ \dots\ ,\ x_k,\ x}$$

Their results are as follows:

$$\mathbb{P}(N_n^{(k)} = x) = \sum_{i=0}^{k-1} \sum_{x_1,\dots,x_k} \binom{x_1 + \cdots + x_k + x}{x_1\ ,\ \dots\ ,\ x_k,\ x} p^n \left(\frac{q}{p}\right)^{x_1+\cdots+x_k} \tag{7}$$

where the inner summation is over all nonnegative integers $x_1, \dots, x_k$ such that the relation $\sum_{j=1}^{k} jx_j + kx + i = n$ is verified,

$$\mathbb{P}(L_n \le k, S_n = r) = p^r q^{n-r} \sum_{i=0}^{k} \sum_{x_1,\dots,x_{k+1}} \binom{x_1 + \cdots + x_{k+1}}{x_1\ ,\ \dots\ ,\ x_{k+1}} \tag{8}$$

where S_n denote the number of heads in the sequence and the inner summation is over all nonnegative integers $x_1, \dots, x_k$ such that $\sum_{j=1}^{k+1} jx_j = n - i$ and such that $\sum_{j=1}^{k+1} x_j = n - r$,

$$\mathbb{P}(L_n \le k | S_n = r) = \frac{1}{C_n^r} \sum_{i=0}^{k} \sum_{x_1,\dots,x_{k+1}} \binom{x_1 + \cdots + x_{k+1}}{x_1\ ,\ \dots\ ,\ x_{k+1}} \tag{9}$$

in the same conditions as (8).

We will end this section by giving a recurrent formula for the distribution of the length of the longest run of heads, due tu Schilling [13]. Unlike the case of a fair coin (4), in the case of a biassed coin (with the probability of heads p) we need to consider, in addition to the length of the sequence, the total number of heads ($S_n = k$). Let the number of strings of length n in which exactly k heads occur and such that no more than m of these occur consecutively be denoted by $C_n^{(k)}(m)$. Then the probability that the length of the longest head run does not exceed m can be expressed as:

$$\mathbb{P}(L_n \le m) = \sum_{k=0}^{n} C_n^{(k)}(m) p^k q^{n-k}$$

Using the same notation as in the case of a fair coin, we can observe that the number of sequences of length n in which the longest run of heads does not exceed m can be written as:

$$A_n(m) = \sum_{k=0}^{n} C_n^{(k)}(m)$$

If we use the same argument of portioning as in the fair coin case we get the following recursion formula:

$$C_n^{(k)}(m) = \begin{cases} \sum_{j=0}^{m} C_{n-j-1}^{(k-j)}(m), & \text{for } n > k > m \\ C_n^m, & \text{for } k \leq m \\ 0 & \text{for } m < k = n. \end{cases} \tag{10}$$

2. The asymptotic behavior of the longest head run

We can notice that the exact formulas, and even the recurrent ones, are not of much help when we deal with long sequences (or the ratio $\frac{n}{m}$ is big). Take for example $n = 100,000$ tosses and look for the probability of getting a run of at least 25 heads of a fair coin, then the exact formula (5) involves a sum of $\left[\frac{n}{m}\right] = 4000$ terms. For the case in which n is large a series of approximations, bounds and asymptotic formulas were developed.

2.1. The case of a fair coin.

First we will consider the case when the probability of head (of success) is $\frac{1}{2}$. Erdös and Révész [5], using a result of Erdös and Rényi [4], were interested in bounds for the length of the longest head run that would hold for almost all $\omega \in \Omega$ (where Ω denotes the basic space) and in some cases they obtained the best possible results.
Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed Bernoulli random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = 0) = \frac{1}{2}$, and let L_n be the length of the longest head run among first n coin tosses, i.e. L_n is the larger integer L such that there is a k, $1 \leq k \leq n - L + 1$, with $X_k = X_{k+1} = \cdots = X_{k+L-1} = 1$. We will give this results below:

Theorem 2.1 (Th. 3 & 4 from [5])**.** *Let α_n a sequence of positive integers. If*

$$\sum_{n=1}^{\infty} 2^{-\alpha_n} = \infty,$$

then for almost all $\omega \in \Omega$ there is an infinite sequence $n_i = n_i(\omega, (\alpha_n)_n)$, $i \geq 1$, of integers such that

$$L_{n_i} \geq \alpha_{n_i}, \quad i \geq 1.$$

If, on the other hand,

$$\sum_{n=1}^{\infty} 2^{-\alpha_n} < \infty,$$

then for almost all $\omega \in \Omega$ there exists a positive integer $n_0 = n_0(\omega, (\alpha_n)_n)$ such that

$$L_n < \alpha_n, \quad \forall n \geq n_0.$$

The following theorem, due to Erdös and Rényi [4], gives bounds for L_n:

Theorem 2.2. *Let $0 < C_1 < 1 < C_2 < \infty$ then for almost all $\omega \in \Omega$ there exists a finite integer $N_0 = N_0(\omega, C_1, C_2)$ such that*

$$[C_1 \log_2 N] \leq L_n \leq [C_2 \log_2 N]$$

if $n \geq N_0$.

Starting from the Theorem 2.2, Erdös and Révész obtained in [5] sharper bounds for L_n. The next theorem illustrates this:

Theorem 2.3 (Th. 1 & 2 from [5])**.** *Let ϵ be any positive number. Then for almost all $\omega \in \Omega$ there exists a finite integer $N_0 = N_0(\omega, \epsilon)$ such that for $n \geq N_0$,*

$$L_n \geq [\log_2 n - \log_2 \log_2 \log_2 n + \log_2 \log_2 e - 2 - \epsilon].$$

Also for almost all $\omega \in \Omega$ there exists an infinite sequence $n_i = n_i(\omega, \epsilon)$, $i \geq 1$, of positive integers such that

$$L_n < [\log_2 n_i - \log_2 \log_2 \log_2 n_i + \log_2 \log_2 e - 1 - \epsilon].$$

We observe that between the bounds given in the Theorem 2.3 there is a gap which is essentially equal to unity. Guibas and Odlyzko gave in [7] sharper bounds for L_n, they also extend and deepen these results to include arbitrary repetitive patterns of heads and tails and that's the reason why we will discuss about their results in the final section. In his article, Turi [14] gave some interesting results regarding the limit behavior of the longest run of pure heads and of pure heads or pure tails . He stated that, under some conditions, the distribution of the number of pure head runs of length k (and also for the pure heads or pure tails), $N_n^{(k)}$, converges to a compound Poisson distribution:

Theorem 2.4 (Th.2.2 in [14])**.** *If $n \to \infty$ and $k \to \infty$ such that*

$$\frac{n}{2^{k+1}} \to \lambda$$

then we have

$$\mathbb{E}(s^{N_n^{(k)}}) \to e^{\lambda\left(\frac{\left(1-\frac{1}{2}\right)z}{1-\frac{1}{2}z}-1\right)}.$$

Theorem 2.5 (Th.2.4 in [14])**.** *For any integer k we have*

$$\mathbb{P}(L_n - [\log_2 n] < k) = e^{-2^{-(k+1-\{\log_2 n\})}} + o(1)$$

where $[a]$ denotes the integer part of a and $\{a\} = a - [a]$.

If we ask about the expectation or the variance of the length of the longest head run, Boyd found some interesting results in [2], using generating functions. He computed the expected maximum run of losses in n trials, denoted by $M(n)$, by the formula:

$$M(n) = \frac{\ln n + \gamma}{\ln 2} - \frac{3}{2} + \epsilon_1\left(\frac{\ln n}{\ln 2}\right) + O\left(\frac{(\ln n)^4}{n}\right) \tag{11}$$

where γ is Euler's constant, $\epsilon_1(\alpha)$ is a function of period 1 which satisfies for all α: $|\epsilon_1(\alpha)| < 2 \times 10^{-6}$. He also found that the variance $Var(L_n)$ is bounded for $n \to \infty$ and the following relation is valid:

$$V(n) = \frac{1}{12} + \frac{\pi^2}{6(\ln 2)^2} + \epsilon_2\left(\frac{\ln n}{\ln 2}\right) + O\left(\frac{(\ln n)^5}{n}\right) \tag{12}$$

where $\epsilon_2(\alpha)$ is a function of period 1 which satisfies for all α: $|\epsilon_2(\alpha)| < 10^{-4}$. We have to observe, however, that the functions $\epsilon_1(\alpha)$, $\epsilon_2(\alpha)$ possesses no limit and that leads to the conclusion that the longest head runs possesses no limit distribution. Note that from the above formula for the variance we can deduce that the standard deviation is approximately $(V(n))^{\frac{1}{2}} \approx \left(\frac{1}{12} + \frac{\pi^2}{6(\ln 2)^2}\right)^{\frac{1}{2}} = 1.873$, a very small value that implies the fact that the length of the longest head run is quite predictable (is within about two of its expectation). In order to prove the results (11) and (12), Boyd showed first that the function $F(n,k)$, the probability that there is no losing run of length greater than k in a sequence of n trials, has an interesting limiting behavior (Th.1 in [2]). He showed that, as $n \to \infty$, uniformly in k:

$$\left|F(n,k) - e^{-2^{-(k+\log_2 n)}}\right| = O\left(\frac{(\ln n)^3}{n}\right) \tag{13}$$

The result is very useful and a generalization, due to Gordon, Schilling and Waterman in [6], will be presented in the next section.

2.2. **The case of a biassed coin.** Now we consider the case in which we have a sequence of Bernoulli trials with probability of head (or success) equal to $p \in (0,1)$. Like in the previous section we try to find bounds and the asymptotic behavior for the length of the longest head run. Let use the notation $R(k;n,p) = \mathbb{P}(L_n < k)$ for the probability that the length of the longest head run is strictly less than k. The following theorem, due to Muselli [9], gives an inequality used subsequent in the paper to determine different bounds for $R(k;n,p)$:

Theorem 2.6 (Th.1 and Cor.1 in [9])**.** *If $1 < k < n$ the following inequalities holds:*

$$R(k;m,p)R(k;n-m+k-1,p) < R(k;n,p) < R(k;m,p)R(k;n-m,p) \tag{14}$$

for every integer m such that $k \leq m < n$ and the left inequality being also true for $0 \leq m < k$.

Noticing that for $m = k$ we have $R(k;k,p) = 1 - p^k$ (replacing $m = k = n$ in equation (5)) and using the inequality (14), we obtain

$$\left(1-p^k\right) R(k;n-1,p) < R(k;n,p) < \left(1-p^k\right) R(k;n-k,p)$$

and by iterating the application of (14), for $m = k$ we obtain the bounds:

$$\left(1-p^k\right)^{n-k+1} < R(k;n,p) < \left(1-p^k\right)^{\left\lfloor \frac{n}{k} \right\rfloor} \tag{15}$$

A stronger lower bound for $R(k;n,p)$, obtained also by application of Theorem 2.6, is the following theorem:

Theorem 2.7 (Th.2 and Cor.2 in [9])**.** *For every $1 \leq k \leq n - h(k,p)$, where $h(k,p) = \left\lfloor \frac{1-p^k}{q} \right\rfloor$ and $l(k,p) = \left\lfloor \frac{n-k}{h(k,p)+1} \right\rfloor$, the following inequalities holds:*

$$R(k;n,p) \geq \left(1-p^k\right)^{n-k+1-l(k,p)(h(k,p)-1)} \tag{16}$$

$$\left(1-p^k\right)^{n-k+1-l(k,p)(h(k,p)-1)} \geq \left(1-p^k\right)^{n-k+1} \tag{17}$$

Referring to the limiting behavior of L_n, Gordon, Schilling and Waterman proved in [6] some results regarding the distribution of the longest k-interrupted head run $L_k(n)$, and they also managed to generalize the results obtained by Boyd in [2]. They analyzed the head runs by using geometric random variables as follows: let $Y_0(1), Y_0(1), \ldots$ be an i.i.d. sequence of geometric random variables with parameter $q = 1-p$, so that $\mathbb{P}\left(Y_0(n) = m\right) = qp^m$, and $S(m) = m + \sum_{j=1}^{m} Y_0(j)$ for $m \geq 1$ and $S(m) = 0$ otherwise. The values of $S(m)$ represent the locations of T's in the sequence $X_1, X_2, \ldots$ of heads and tails, where $X_n = \mathbf{1}_{\{n \neq S(j),\ \forall j > 0\}}$. It is easy to see that $Y_0(m)$ represents the length of the m-th completed pure head run, which at the $S(m)$ toss ends by a T. If we write $M_0(n) = \max_{m \leq n} Y_0(m)$, and we take $N(n)$ to represent the number of tails in the firs n tosses, then the length of the longest head run is defined by $L_n = \max\{M_0(N(n)), n - S(N(n))\}$. We have to observe that we used only the subscript 0 because the general treatment will be given in the last part. Their main results (for $k = 0$) are:

Theorem 2.8 (Th.1 in [6]). *Let W have an extreme standard distribution, i.e. $\mathbb{P}(W \leq x) = e^{-e^{-x}}$, and $\delta_0(n) = \ln\left(\frac{1}{p}\right) - \left\lfloor \ln\left(\frac{1}{p}\right)\right\rfloor$. Then, uniformly in t,*

$$\mathbb{P}(L_n - \log_{\frac{1}{p}}(qn) \leq t) - \mathbb{P}\left(\left\lfloor \frac{W}{\ln\left(\frac{1}{p}\right)} + \delta_0(qn)\right\rfloor - \delta_0(qn) \leq t\right) \to 0$$

as $n \to \infty$.

which shows that L_n is well approximated by $M_0(N(n))$. Versions of next theorem can be found also in Boyd [2] and in Guibas and Odlyzko [7]:

Theorem 2.9 (Th.2 in [6]). *Let $\theta = \frac{\pi^2}{\ln\left(\frac{1}{p}\right)}$. Then*

$$\left|\mathbb{E}(L_n) - \left(\log_{\frac{1}{p}}(qn) + \frac{\gamma}{\ln\left(\frac{1}{p}\right)} - \frac{1}{2}\right)\right| < \frac{\theta^{\frac{1}{2}}}{2\pi e^{\theta}\left(1 - e^{-\theta}\right)^2} + o(1) \tag{18}$$

and

$$\left|Var(L_n) - \left(\frac{\pi^2}{6\left(\ln\left(\frac{1}{p}\right)\right)^2} + \frac{1}{12}\right)\right| < \frac{2(1.1 + 0.7\theta)\theta^{\frac{1}{2}}}{2\pi e^{\theta}\left(1 - e^{-\theta}\right)^3} + o(1). \tag{19}$$

It can be seen that the results obtained in Theorem 2.8 and in Theorem 2.9 are in close relation with the equations (11) and (12) obtained by Boyd in [2]. We will finish this section by characterizing the limiting behavior for the probability that the longest run is formed by heads (successes). We notate this probability by $V_n(p)$ and we notice that $V_n(p) = \mathbb{P}(L_n > \bar{L_n})$, where $\bar{L_n}$ represents the length of the longest failure (tail) run in the n trials.

Theorem 2.10 (Th.5 in [9]). *The probability $V_n(p)$ has the following limit values*

$$\lim_{n \to \infty} V_n(p) = \begin{cases} 0, & \text{if } 0 \leq p < \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < p \leq 1. \end{cases} \tag{20}$$

The Theorem 2.10 can be interpreted as follows: let A and B be two players that toss a biassed coin that give a good outcome for the first player, i.e. for example heads with probability p, and a good outcome for the second player as tails with probability $q = 1-p$; and consider that after n tosses wins the player who has scored the longest sequence of favorable outcomes. Now if the number of tosses increase indefinitely then the player which has the greater probability of getting favorable outcomes will win for sure.

3. Extensions and further results

As we announced in the previous sections there are many other types of run phenomena related to the length of the longest run. One of these phenomena that we will describe in what follows is the longest k-interrupted head run $L_k(n)$. Erdös and Révész studied this problem and showed in [5] (see also Révész [12]) some limiting properties (which are generalizations of the Theorem 2.1 and Theorem 2.3) for the longest k-interrupted head run, but only in the case of fair coins. Other generalizations can be found also in Guibas and Odlyzko [7] and in Gordon, Schilling and Waterman in [6] (for example generalizations for the Theorem 2.8 and Theorem 2.9). We will give these results below, but let's do some notations first. Consider that we are in the situation described before the Theorem 2.8 and consider now the case of the head runs interrupted by kT's. Denote the summed lengths of $k+1$ pure head run starting with run m and ending with run $m+k$, plus the kT's that separate the $k+1$ component head runs, by $Y_k(m) = S(m+k) - (1 + S(m-1))$, and note that the random variable $Y_k(m) - k = \sum_{j=0}^{k} Y_0(m+j)$ has negative binomial distribution with parameters $(k+1, q)$. Write $M_k(n) = \max_{m \le n} Y_k(m)$, when $n > 0$ and 0 otherwise and define the length of the longest k-interrupted head run in the first n tosses by $L_k(n) = \max\{M_k(N(n)-k), n - S(N(n)-k)\}$ where, as before, $N(n)$ represents the number of tails in the firs n tosses. Letting $\lambda = \ln\left(\frac{1}{p}\right)$ and $\mu_k(n) = \log_{\frac{1}{p}}(n) + k\log_{\frac{1}{p}}\log_{\frac{1}{p}}(n) + k\log_{\frac{1}{p}}\left(\frac{q}{p}\right) - \log_{\frac{1}{p}}(k!)$ we have:

Theorem 3.1 (generalization of Theorem 2.8)**.** *Let W have an extreme standard distribution, i.e. $\mathbb{P}(W \le x) = e^{-e^{-x}}$, and $\delta_k(n) = \mu_k(n) - \lfloor \mu_k(n) \rfloor$. Then, uniformly in t,*

$$\mathbb{P}(L_k(n) - \mu_k(qn) \le t) - \mathbb{P}\left(\left\lfloor \frac{W}{\lambda} + \delta_k(qn) \right\rfloor - \delta_k(qn) \le t\right) \to 0$$

as $n \to \infty$.

Theorem 3.2 (generalization of Theorem 2.9)**.** *Let* $\theta = \frac{\pi^2}{\lambda}$*. Then*

$$\left|\mathbb{E}(L_k(n)) - \left(\mu_k(qn) + \frac{\gamma}{\lambda} - \frac{1}{2}\right)\right| < \frac{\theta^{\frac{1}{2}}}{2\pi e^{\theta}\left(1-e^{-\theta}\right)^2} + o(1) \tag{21}$$

and

$$\left|Var(L_k(n)) - \left(\frac{\pi^2}{6\lambda^2} + \frac{1}{12}\right)\right| < \frac{2(1.1+0.7\theta)\theta^{\frac{1}{2}}}{2\pi e^{\theta}\left(1-e^{-\theta}\right)^3} + o(1). \tag{22}$$

Regarding the almost sure behavior of the longest k-interrupted head run, Guibas and Odlyzko in [7] (by means of generating functions in theorems $\star 3$ and $\star 3$) and Gordon, Schilling and Waterman in [6], presented a complete characterization of the function which are touched infinitely often by usually long or short longest head runs. We will remark that the following theorems are generalizations of the results obtained by Erdös and Révész in [5] (Theorem 2.1):

Theorem 3.3 (Th.3 in [6])**.** *Let* α_n *a non-decreasing sequence of integers. Then*

$$\mathbb{P}(L_k(n) \geq \alpha_n \ i.o.) = \begin{cases} 0, & if \sum \alpha_n^k p^{\alpha_n} < \infty \\ 1, & if \sum \alpha_n^k p^{\alpha_n} = \infty. \end{cases} \tag{23}$$

A similar theorem, but for $\mathbb{P}(L_k(n) < \alpha_n \ i.o.)$ is given in [6], but require more care and is based on the construction of a sequence of countinous random variables,$\hat{Y_k(j)}$, with a given distribution function and such that the $\max_{j\leq n} \hat{Y_k(j)} + k - \mu_k(qn) \to \frac{W}{\lambda}$ and $Y_k(j) = \left\lfloor \hat{Y_k(j)} \right\rfloor + k$.
We will finish this section by giving a result due to Mori [10] that is in close relation with the results described by Gordon, Schilling and Waterman [6]:

Theorem 3.4 (Cor.5.1 in [10])**.** *The following relation is valid almost sure:*

$$\lim_{n\to\infty} \frac{1}{\ln(n)} \sum_{i=1}^{n} \frac{1}{i} \mathbf{1}_{\left\{L_i(n) - \log_{\frac{1}{p}}(i) - k\log_{\frac{1}{p}}\log_{\frac{1}{p}}(i) < t\right\}} = \int_t^{t+1} e^{-\frac{q}{k!}\left(\frac{q}{p}\right)^k p^z}\, dz. \tag{24}$$

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