Approximations for two-dimensional discrete scan statistics in some dependent models

Alexandru Amărioarei
Cristian Preda

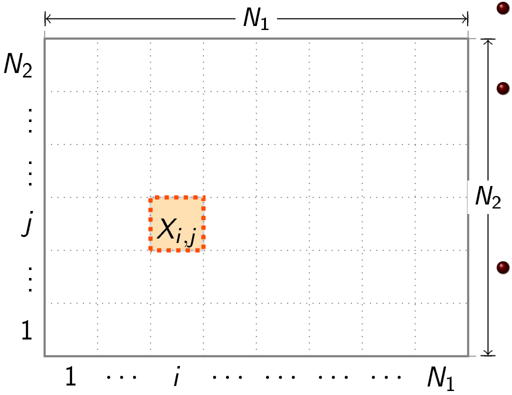
Laboratoire de Mathématiques Paul Painlevé
Département de Probabilités et Statistique
Université de Lille 1, INRIA Modal Team

IMS-China International Conference on Statistics and Probability
30 June - 4 July, 2013, Chengdu, China

Outline

Outline

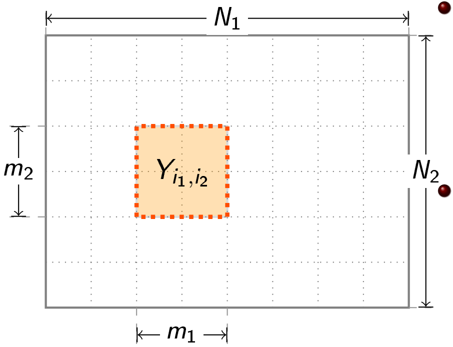
The Two-Dimensional Discrete Scan Statistic



Let N_1, N_2 be positive integers

- Rectangular region $\mathcal{R} = [0, N_1] \times [0, N_2]$
- $(X_{ij})_{\substack{1 \leq i \leq N_1 \\ 1 \leq j \leq N_2}}$ integer r.v.'s
  - Bernoulli($\mathcal{B}(1, p)$)
  - Binomial($\mathcal{B}(n, p)$)
  - Poisson($\mathcal{P}(\lambda)$)
- X_{ij} number of observed events in the elementary subregion $r_{ij} = [i-1, i] \times [j-1, j]$

The Two-Dimensional Discrete Scan Statistic



Let m_1, m_2 be positive integers

- Define for $1 \leq i_j \leq N_j - m_j + 1$,

$$Y_{i_1 i_2} = \sum_{i=i_1}^{i_1+m_1-1} \sum_{j=i_2}^{i_2+m_2-1} X_{ij}$$

- The two dimensional scan statistic,

$$S_{m_1,m_2}(N_1, N_2) = \max_{\substack{1 \leq i_1 \leq N_1 - m_1 + 1 \\ 1 \leq i_2 \leq N_2 - m_2 + 1}} Y_{i_1 i_2}$$

- Used for testing the null hypotheses of randomness against the alternative hypothesis of clustering

Problem and related results

Problem

Approximate the distribution of two dimensional discrete scan statistic

$$\mathbb{P}\left(S_{m_1,m_2}(N_1,N_2) \leq n\right).$$

- The i.i.d. model:
  - No exact formulas
  - For Bernoulli case:
    - product type approximations (Boutsikas and Koutras 2000)
    - Poisson approximations (Chen and Glaz 1996)
    - bounds (Boutsikas and Koutras 2003)
  - For binomial and Poisson cases: (Glaz 2009)
    - Product type approximation
    - Lower bound
  - Approximation and error bounds (Haiman 2006)
- Dependent model: **no results !**

Outline

Introducing the Model

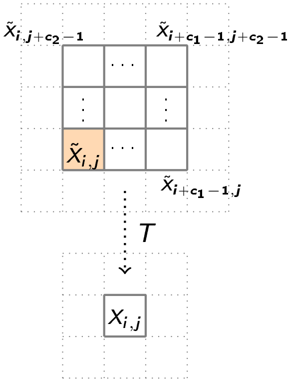
Let $1 \leq c_s \leq \tilde{N}_s$, $s \in \{1,2\}$ integers

- $\left(\tilde{X}_{ij}\right)_{\substack{1\leq i\leq \tilde{N}_1 \\ 1\leq j\leq \tilde{N}_2}}$ i.i.d. r.v.'s
- configuration matrix in (i,j)

$$C_{(i,j)} = \left(C_{(i,j)}(k,l)\right)_{\substack{1\leq k\leq c_2 \\ 1\leq l\leq c_1}}$$

$$C_{(i,j)}(k,l) = \tilde{X}_{i+l-1,j+c_2-k}$$

- transformation $T : \mathcal{M}_{c_2,c_1}(\mathbb{R}) \to \mathbb{R}$

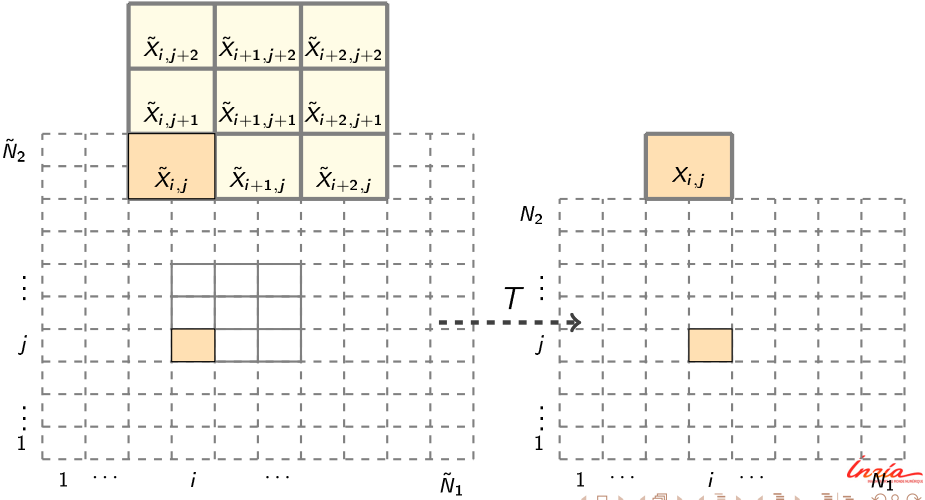


Define the block-factor model, $N_1 = \tilde{N}_1 - c_1 + 1$, $N_2 = \tilde{N}_2 - c_2 + 1$

$$X_{i,j} = T\left(C_{(i,j)}\right), \ \substack{1\leq i\leq N_1 \\ 1\leq j\leq N_2}$$

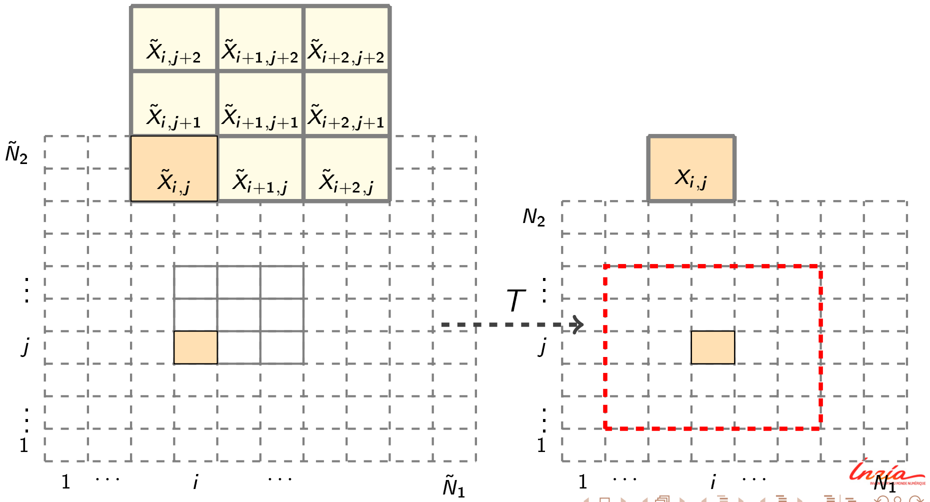
Model: case $c_1 = c_2 = 3$

- To simplify the presentation we consider $c_1 = c_2 = 3$

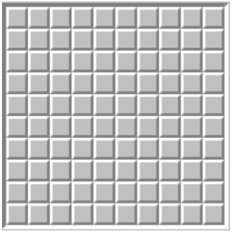


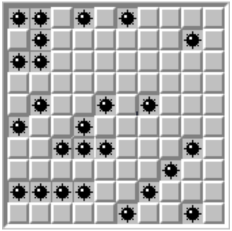
Model: case $c_1 = c_2 = 3$

- To simplify the presentation we consider $c_1 = c_2 = 3$



Example: A game of minesweeper





Model:

- $\tilde{X}_{i,j} \sim \mathcal{B}(p)$ (presence, absence of a mine)
- number of neighboring mines

$$T\left(C_{(i,j)}\right) = \sum_{\substack{(s,t)\in\{0,1,2\}^2 \\ (s,t)\neq(1,1)}} \tilde{X}_{i+s,j+t}$$

- $X_{i,j} = T\left(C_{(i,j)}\right)$

Outline

Key Idea

Haiman(2000) proposed a different approach

Main Observation

The scan statistic r.v. can be viewed as a maximum of a sequence of 1-dependent stationary r.v..

- The idea:
  - discrete and continuous one dimensional scan statistic: Haiman (2000,2007)
  - discrete and continuous two dimensional scan statistic: Haiman and Preda (2002,2006)
  - discrete three dimensional scan statistic: Amarioarei (2013)

Writing the Scan as an Extreme of 1-Dependent R.V.'s

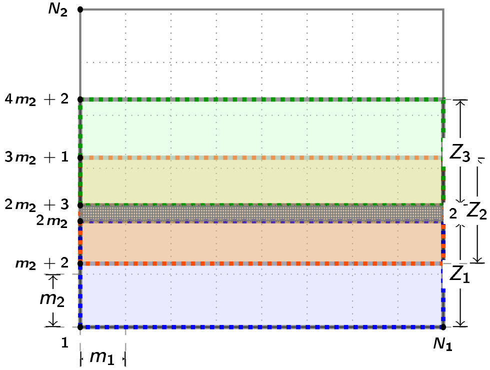
Let $N_j = (L_j + 1)(m_j + 1) - 2$, $j \in \{1, 2\}$ positive integers

- Define for $l \in \{1, 2, \ldots, L_2\}$

$$Z_l = \max_{\substack{1 \le i_1 \le L_1(m_1+1) \\ (l-1)(m_2+1)+1 \le i_2 \le l(m_2+1)}} Y_{i_1 i_2}$$

- $(Z_l)_l$ is 1-dependent and stationary
- Observe

$$S_{m_1,m_2}(N_1, N_2) = \max_{1 \le l \le L_2} Z_l$$



Main Tool

Let $(Z_j)_{j\geq 1}$ be a strictly stationary 1-dependent sequence of r.v.'s and let $q_m = q_m(x) = \mathbb{P}(\max(Z_1,\dots,Z_m)\leq x)$, with $x < \sup\{u|\mathbb{P}(Z_1\leq u)<1\}$.

Main Theorem (Haiman 1999, Amarioarei 2012)

For x such that $\mathbb{P}(Z_1 > x) = 1-q_1 \leq \alpha < 0.1$ and $m > 3$ we have

$$\left|q_m - \frac{6(q_1-q_2)^2+4q_3-3q_4}{(1+q_1-q_2+q_3-q_4+2q_1^2+3q_2^2-5q_1q_2)^m}\right| \leq \Delta_1(1-q_1)^3,$$

$$\left|q_m - \frac{2q_1-q_2}{[1+q_1-q_2+2(q_1-q_2)^2]^m}\right| \leq \Delta_2(1-q_1)^2,$$

- $\Delta_1 = \Delta_1(\alpha, q_1, m) = \Gamma(\alpha) + mK(\alpha)$
- $\Delta_2 = mE(\alpha, q_1, m) = m\left[1+\frac{3}{m}+K(\alpha)(1-q_1)+\frac{\Gamma(\alpha)(1-q_1)}{m}\right]$.

Selected values for $K(\alpha)$ and $\Gamma(\alpha)$

Outline

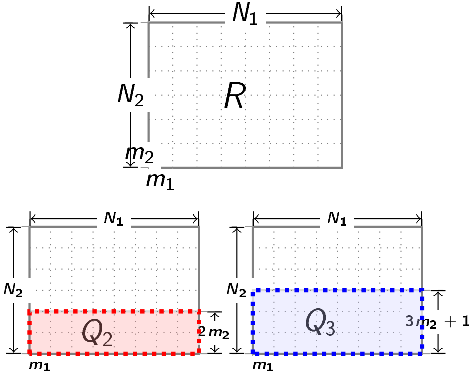
First Step Approximation

Using Main Theorem we obtain

- Define
$$Q_2 = \mathbb{P}(Z_1 \leq k)$$
$$Q_3 = \mathbb{P}(Z_1 \leq k, Z_2 \leq k)$$

- If $1 - Q_2 \leq \alpha_1 < 0.1$ the (first) approximation
$$\mathbb{P}(S \leq k) \approx \frac{2Q_2 - Q_3}{[1 + Q_2 - Q_3 + 2(Q_2 - Q_3)^2]^{L_2}}$$
where $S = S_{m_1,m_2}(N_1, N_2)$

- Approximation error
$$L_2 E(\alpha_1, L_2)(1 - Q_2)^2$$



Second Step Approximation

$\underline{Q_2}$:

- For $s \in \{1, 2, \ldots, L_1\}$

$$Z_s^{(2)} = \max_{\substack{(s-1)(m_1+1)+1 \leq i_1 \leq s(m_1+1) \\ 1 \leq i_2 \leq m_2+1}} Y_{i_1 i_2}$$

- $$Q_2 = \mathbb{P}\left(\max_{1 \leq s \leq L_1} Z_s^{(2)} \leq k\right)$$

- Define

$$Q_{22} = \mathbb{P}(Z_1^{(2)} \leq k)$$

$$Q_{32} = \mathbb{P}(Z_1^{(2)} \leq k, Z_2^{(2)} \leq k)$$

- Approximation $(1 - Q_{22} \leq \alpha_2)$

$$Q_2 \approx \frac{2Q_{22} - Q_{32}}{\left[1 + Q_{22} - Q_{32} + 2(Q_{22} - Q_{32})^2\right]^{L_1}}$$

- Error

$$L_1 E(\alpha_2, L_1)(1 - Q_{22})^2$$

$\underline{Q_3}$:

- For $s \in \{1, 2, \ldots, L_1\}$

$$Z_s^{(3)} = \max_{\substack{(s-1)(m_1+1)+1 \leq i_1 \leq s(m_1+1) \\ 1 \leq i_2 \leq 2(m_2+1)}} Y_{i_1 i_2}$$

- $$Q_3 = \mathbb{P}\left(\max_{1 \leq l \leq L_1} Z_s^{(3)} \leq k\right)$$

- Define

$$Q_{23} = \mathbb{P}(Z_1^{(3)} \leq k)$$

$$Q_{33} = \mathbb{P}(Z_1^{(3)} \leq k, Z_2^{(3)} \leq k)$$

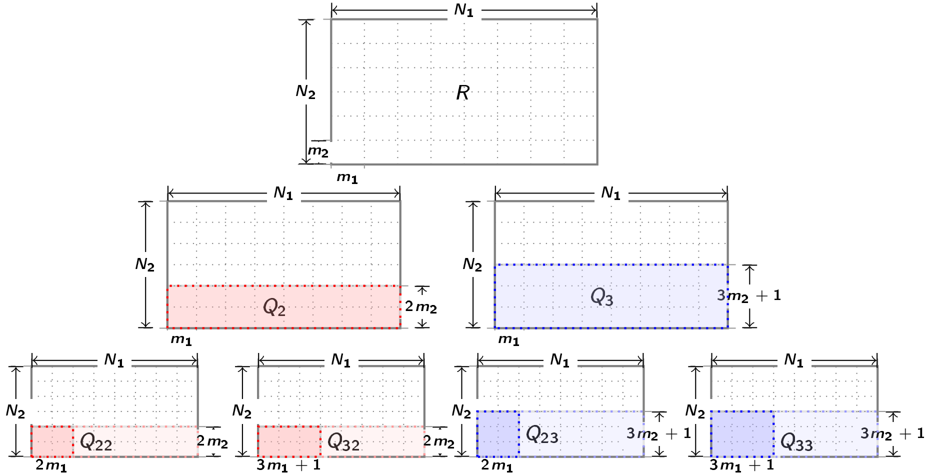
- Approximation $(1 - Q_{23} \leq \alpha_2)$

$$Q_3 \approx \frac{2Q_{23} - Q_{33}}{\left[1 + Q_{23} - Q_{33} + 2(Q_{23} - Q_{33})^2\right]^{L_1}}$$

- Error

$$L_1 E(\alpha_2, L_1)(1 - Q_{23})^2$$

Illustration of the Approximation Process



Outline

Theoretical Approximation Error

Define for $s \in \{2, 3\}$

$$H(x, y, m) = \frac{2x - y}{[1 + x - y + 2(x - y)^2]^m}, \ \alpha_1 = 1 - Q_3, \ \alpha_2 = 1 - Q_{23},$$
$$F_1 = E(\alpha_2, L_1), \ F_2 = E(\alpha_1, L_2), \ R_s = H\left(Q_{2s}, Q_{3s}, L_1\right),$$

The approximation error

$$\boxed{E_{app} = L_2 F_2 B_2^2 + L_1 L_2 F_1 \left[(1 - Q_{22})^2 + (1 - Q_{23})^2\right]}$$

where B_2 is given by

$$\boxed{B_2 = 1 - R_2 + L_1 F_1 (1 - Q_{22})^2}$$

Outline

Simulation Error for Approximation Formula

If *ITER* is the number of simulations, we can say, at 95% confidence level,

$$\left|Q_{rt} - \hat{Q}_{rt}\right| \leq 1.96\sqrt{\frac{\hat{Q}_{rt}(1-\hat{Q}_{rt})}{ITER}} = \beta_{rt}, \;\; r, t \in \{2, 3\}$$

where $\hat{Q}_{rt}$ is the simulated value.
Define for $r \in \{2, 3\}$,

$$\hat{Q}_r = H\left(\hat{Q}_{2r}, \hat{Q}_{3r}, L_1\right)$$

The simulation error corresponding to the approximation formula

$$E_{sf} = L_1 L_2 \left(\beta_{22} + \beta_{23} + \beta_{32} + \beta_{33}\right)$$

Simulation Error for Approximation Error

Introducing

$$C_{2r} = 1 - \hat{Q}_{2r} + \beta_{2r}, \quad r \in \{2,3\},$$
$$C_2 = 1 - \hat{Q}_2 + L_1(\beta_{22} + \beta_{32}) + L_1 F_1 C_{22}^2,$$

The simulation error corresponding to the approximation

$$E_{sapp} = L_2 F_2 C_2^2 + L_1 L_2 F_1 \left[C_{22}^2 + C_{23}^2\right]$$

The total error

$$E_{total} = E_{app} + E_{sf} + E_{sapp}$$

Outline

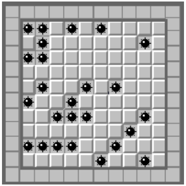
A Game of Minesweeper - Part 2

Recall the model:

- $\tilde{X}_{i,j} \sim \mathcal{B}(p)$ i.i.d. representing the absence/presence of a mine
- $X_{i,j}$ - number of neighboring mines corresponding to (i,j)

$$X_{i,j} = T\left(C_{(i,j)}\right) = \sum_{\substack{(s,t)\in\{0,1,2\}^2 \\ (s,t)\neq(1,1)}} \tilde{X}_{i+s,j+t}$$

$\underline{\tilde{X}_{i,j}}$:



$\xrightarrow{T}$

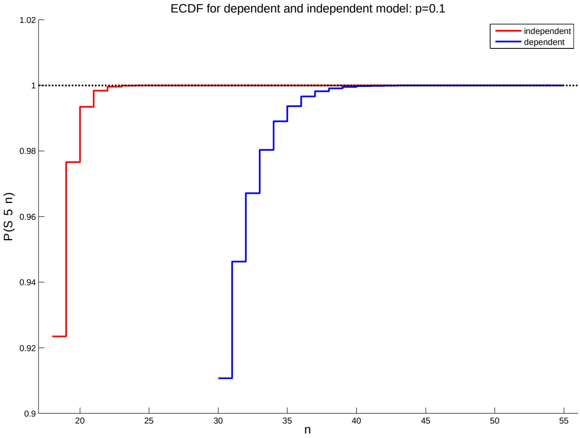
$\underline{X_{i,j}}$:

Numerical Results for $\mathbb{P}(S_{m_1,m_2}(N_1, N_2) \leq n)$

Table 1 : $m_1 = 3, m_2 = 3, N_1 = 42, N_2 = 42, \mathbf{p} = \mathbf{0.1}, ITER = 10^7$

n	Sim Dep	Approx Dep	E_{app}	E_{sim}	E_{total}	Sim Indep	Approx Indep
30	0.88289	0.91068	0.00280	0.01489	0.01770	1	1
31	0.92769	0.94628	0.00085	0.00999	0.01084	1	1
32	0.95632	0.96713	0.00027	0.00725	0.00753	1	1
33	0.97356	0.98033	0.00009	0.00544	0.00553	1	1
34	0.98516	0.98909	0.00002	0.00396	0.00399	1	1
35	0.99161	0.99366	0.00000	0.00298	0.00299	1	1
36	0.99548	0.99663	0.00000	0.00216	0.00216	1	1
37	0.99760	0.99825	0.00000	0.00157	0.00157	1	1
38	0.99864	0.99911	0.00000	0.00110	0.00110	1	1
39	0.99926	0.99955	0.00000	0.00080	0.00080	1	1
40	0.99963	0.99978	0.00000	0.00056	0.00056	1	1
41	0.99987	0.99989	0.00000	0.00037	0.00037	1	1
42	0.99996	0.99994	0.00000	0.00023	0.00023	1	1
43	0.99998	0.99997	0.00000	0.00016	0.00016	1	1
44	0.99998	0.99999	0.00000	0.00009	0.00009	1	1
45	0.99999	0.99999	0.00000	0.00005	0.00005	1	1
46	0.99999	0.99999	0.00000	0.00003	0.00003	1	1

Graphical Illustration: $p = 0.1$

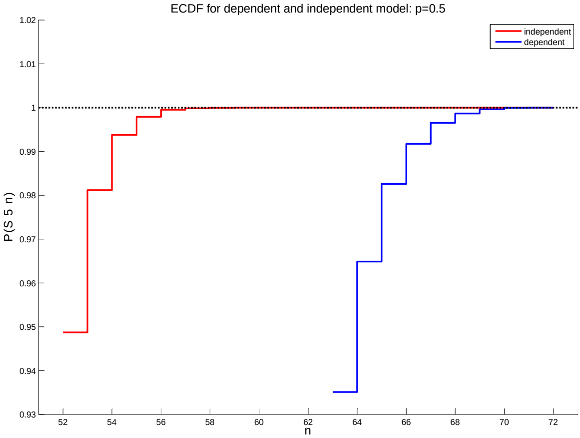


Numerical Results for $\mathbb{P}(S_{m_1,m_2}(N_1, N_2) \leq n)$

Table 2 : $m_1 = 3, m_2 = 3, N_1 = 42, N_2 = 42, \mathbf{p} = \mathbf{0.5}, ITER = 10^7$

n	*Sim Dep*	*Approx Dep*	E_{app}	E_{sim}	E_{total}	*Sim Indep*	*Approx Indep*
62	0.82484	0.88863	0.00487	0.01859	0.02346	1	1
63	0.89706	0.93509	0.00132	0.01139	0.01272	1	1
64	0.94327	0.96484	0.00032	0.00751	0.00784	1	1
65	0.97135	0.98256	0.00007	0.00510	0.00517	1	1
66	0.98668	0.99173	0.00001	0.00339	0.00340	1	1
67	0.99426	0.99650	0.00000	0.00222	0.00222	1	1
68	0.99796	0.99865	0.00000	0.00136	0.00136	1	1
69	0.99929	0.99958	0.00000	0.00077	0.00077	1	1
70	0.99979	0.99992	0.00000	0.00034	0.00034	1	1
71	0.99995	0.99998	0.00000	0.00017	0.00017	1	1
72	1	1	0.00000	0.00000	0.00000	1	1

Graphical Illustration: $p = 0.5$

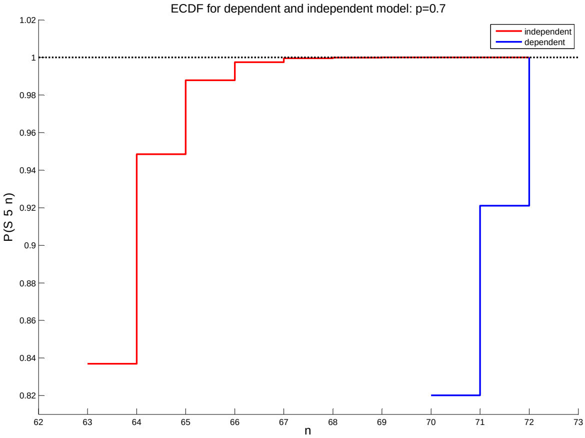


Numerical Results for $\mathbb{P}(S_{m_1,m_2}(N_1, N_2) \leq n)$

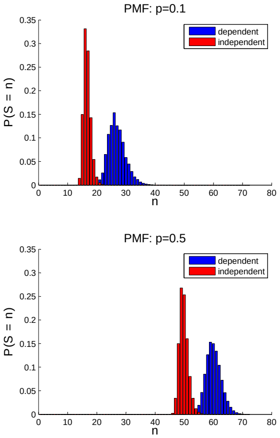
Table 3 : $m_1 = 3, m_2 = 3, N_1 = 42, N_2 = 42, \mathbf{p} = \mathbf{0.7}, ITER = 10^7$

n	*Sim Dep*	*Approx Dep*	E_{app}	E_{sim}	E_{total}	*Sim Indep*	*Approx Indep*
70	0.73026	0.82012	0.01490	0.03271	0.04761	1	0.99999
71	0.87721	0.92103	0.00194	0.01291	0.01485	1	1
72	1	1	0.00000	0.00000	0.00000	1	1

Graphical Illustration: $p = 0.7$



Dependence Effect



- Glaz, J., Naus, J., Wallenstein, S.: Scan statistic. *Springer* (2001).
- Glaz, J., Pozdnyakov, V., Wallenstein, S.: Scan statistic: Methods and Applications. *Birkhauser* (2009).
- Amarioarei, A.: *Approximation for the distribution of extremes of one dependent stationary sequences of random variables*, arXiv:1211.5456v1 (submitted)
- Amarioarei, A.: *Approximation for the Distribution of Three-dimensional Discrete Scan Statistic*, rXiv:1303.3775 (submitted)
- Boutsikas, M.V., Koutras, M.: Reliability approximations for Markov chain imbeddable systems. *Methodol Comput Appl Probab* **2** (2000), 393–412.
- Boutsikas, M. and Koutras, M. *Bounds for the distribution of two dimensional binary scan statistics*, Probability in the Engineering and Information Sciences, 17, 509–525, 2003.

- Chen, J. and Glaz, J. *Two-dimensional discrete scan statistics*, Statistics and Probability Letters 31, 59–68, 1996.
- Haiman, G.: First passage time for some stationary sequence. *Stoch Proc Appl* **80** (1999), 231–248.
- Haiman, G.: Estimating the distribution of scan statistics with high precision. *Extremes* **3** (2000), 349–361.
- Haiman, G., Preda, C.: A new method for estimating the distribution of scan statistics for a two-dimensional Poisson process. *Methodol Comput Appl Probab* **4** (2002), 393–407.
- Haiman, G., Preda, C.: Estimation for the distribution of two-dimensional scan statistics. *Methodol Comput Appl Probab* **8** (2006), 373–381.
- Haiman, G.: Estimating the distribution of one-dimensional discrete scan statistics viewed as extremes of 1-dependent stationary sequences. *J. Stat Plan Infer* **137** (2007), 821–828.

Selected Values for $K(\alpha)$ and $\Gamma(\alpha)$

α	$K(\alpha)$	$\Gamma(\alpha)$
0.1	38.63	480.69
0.05	21.28	180.53
0.025	17.56	145.20
0.01	15.92	131.43

Table 4 : Selected values for $K(\alpha)$ and $\Gamma(\alpha)$

Return